Tame geometry and Hodge theory

1/ Hodge theory : a linearization of complex algebraic geometry

- X nice topological space

$$\xrightarrow[(1895)]{\text{Poincaré}} \{ H_i(X,\mathbb{Z}) = \mathbb{Z}^{b_i(X)} \oplus \text{finite abelian group} \}_{1 \le i \le n} \quad \text{homology groups}$$

Homology is a coarse invariant : it is homotopy invariant.

- X oriented smooth manifold

$$\xrightarrow[(1931)]{\text{De Rham}} \quad \begin{array}{c} [H_i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}]^{\vee} \\ \| \\ H^i(X,\mathbb{Z}) \otimes \mathbb{C} \simeq H^i(\underbrace{A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots \xrightarrow{d} A^n(X)}_{\text{De Rham complex of differential forms}}) =: H^i_{dR}(X,\mathbb{C}) \end{array}$$

Explicitly :

$$H_i(X,\mathbb{Z}) \times H^i_{dR}(X,\mathbb{C}) \longrightarrow \mathbb{C}$$

$$([Z], [\alpha]) \longmapsto \int_Z \alpha$$

• X smooth complex projective variety

$$X = \{[z_0, \ldots, z_n] \in \mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^\times \ , \ f_1(z_0, \ldots, z_n) = \cdots = f_r(z_0, \ldots, z_n) = 0\}$$

for $f_i \in \mathbb{C}[z_0, \ldots, z_n]$ homogeneous

Hodge (1941) $\quad H^k(X, \mathbb{C}) = H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$

$$H^{p,q}(X) = \{[\omega] \in H^k(X, \mathbb{C});\ \omega \in A^{p,q}(X),\ d\omega = 0\}$$

Algebraic interpretation: $H^k_{dR}(X, \mathbb{C}) \underset{can}{\simeq} \mathbb{H}^k(X/\mathbb{C}, \Omega^\bullet_{X/\mathbb{C}})$ — algebraic De Rham complex

$$F^p := \bigoplus_{r \geq p} H^{r, k-r} \simeq \mathbb{H}^k(X/\mathbb{C}, \Omega^{\bullet \geq p}_{X/\mathbb{C}})$$

$H^k(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ has a canonical filtration = the Hodge filtration $F^\bullet$, which varies algebraically with X

. The rational structure $H^k(X,\mathbb{Q})$ on the topological side and the Hodge filtration $F^\cdot$ on the algebraic side interact in a subtle way:

- The cohomology $H^\cdot(X,\mathbb{Q})$ is a rough invariant for $\mathcal{C}^\infty$-manifolds.

  On the other hand the pair $(H^\cdot(X,\mathbb{Q}), F^\cdot)$ is a fine invariant, often sufficient to characterize X (Torelli problem)

- while, for X a smooth manifold, any class in $H^k(X,\mathbb{Q})$ is a $\mathbb{Q}$-linear combination of cycle classes of smooth codimension k submanifolds,

  for $X/\mathbb{C}$ algebraic, the cycle class of a (complex) codimension k algebraic subvariety lies in $H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$.

Hodge conjecture: any class in $H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$ is algebraic.

2/ The Hodge isomorphism is transcendental

$$\begin{tikzcd} \text{Suppose} \quad X = \mathcal{X}_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f" ', "\text{smooth projective}"] \\ \{0\} \arrow[r, hook] & S_{/\mathbb{C}} \end{tikzcd}$$

- Locally: On a sufficiently small neighbourhood $U \ni \{0\}$: $\mathcal{X}_{|U} \simeq X \times U$ which gives a canonical trivialization $\{H^k(\mathcal{X}_s, \mathbb{Q})\}_{s \in U} \simeq H^k(X, \mathbb{Q}) \times U$

  The filtration $F^\bullet$ on $H^k(\mathcal{X}_s, \mathbb{C})$ is thus described by

$$\begin{aligned} \underline{\Phi} : U &\longrightarrow \text{flag variety } \check{D} \text{ for } H^k(X, \mathbb{C}) \\ s &\longmapsto F_s^\bullet \end{aligned}$$

This map is holomorphic, and satisfies a differential system:

$$\frac{\partial}{\partial s} F_s^\bullet \subset F_s^{\bullet - 1}$$

• Globally: $\big(\mathbb{V} = R^k f_* \mathbb{Z}$, $(\mathcal{V} = R^k f_* \Omega^\bullet_{\mathcal{X}/S}, F^\bullet, \nabla)$, polarisation$\big)$ $\mathbb{Z}$VHS

local system on S^{an} with fiber $H^k(\mathcal{X}_s, \mathbb{Z})$

$$\nabla F^\bullet \subset F^{\bullet-1} \otimes \Omega^1_S$$

$$\begin{cases} \Phi : S \longrightarrow \Gamma\backslash D & \text{period map, holomorphic} \\ d\Phi(TS) \subset T_h(\Gamma\backslash D) \end{cases}$$

Example: $\mathcal{M}_g$ = moduli of smooth projective curves of genus g, $\mathcal{C} \to \mathcal{M}_g$ universal curve

$$\Phi : \mathcal{M}_g \longrightarrow \Gamma\backslash\mathbb{H}_g$$

$$[C_s] \longmapsto [\underset{g}{H^{1,0}(C_s)} \subset \underset{2g}{H^1(C, \mathbb{C})}]$$

Remarkable in this example: • $\Gamma\backslash\mathbb{H}_g$ has a natural complex algebraic structure

$\quad = \mathcal{A}_g$ moduli space of p.p. Abelian varieties

• $\Phi : \mathcal{M}_g \hookrightarrow \mathcal{A}_g$ is algebraic!

weight 1

In general: $\Gamma \backslash D$ is a complex manifold with no algebraic structure, usually not compact.

How wild is the $\mathbb{C}$-analytic period map $\Phi : S \longrightarrow \Gamma \backslash D$?

3/ Tame geometry

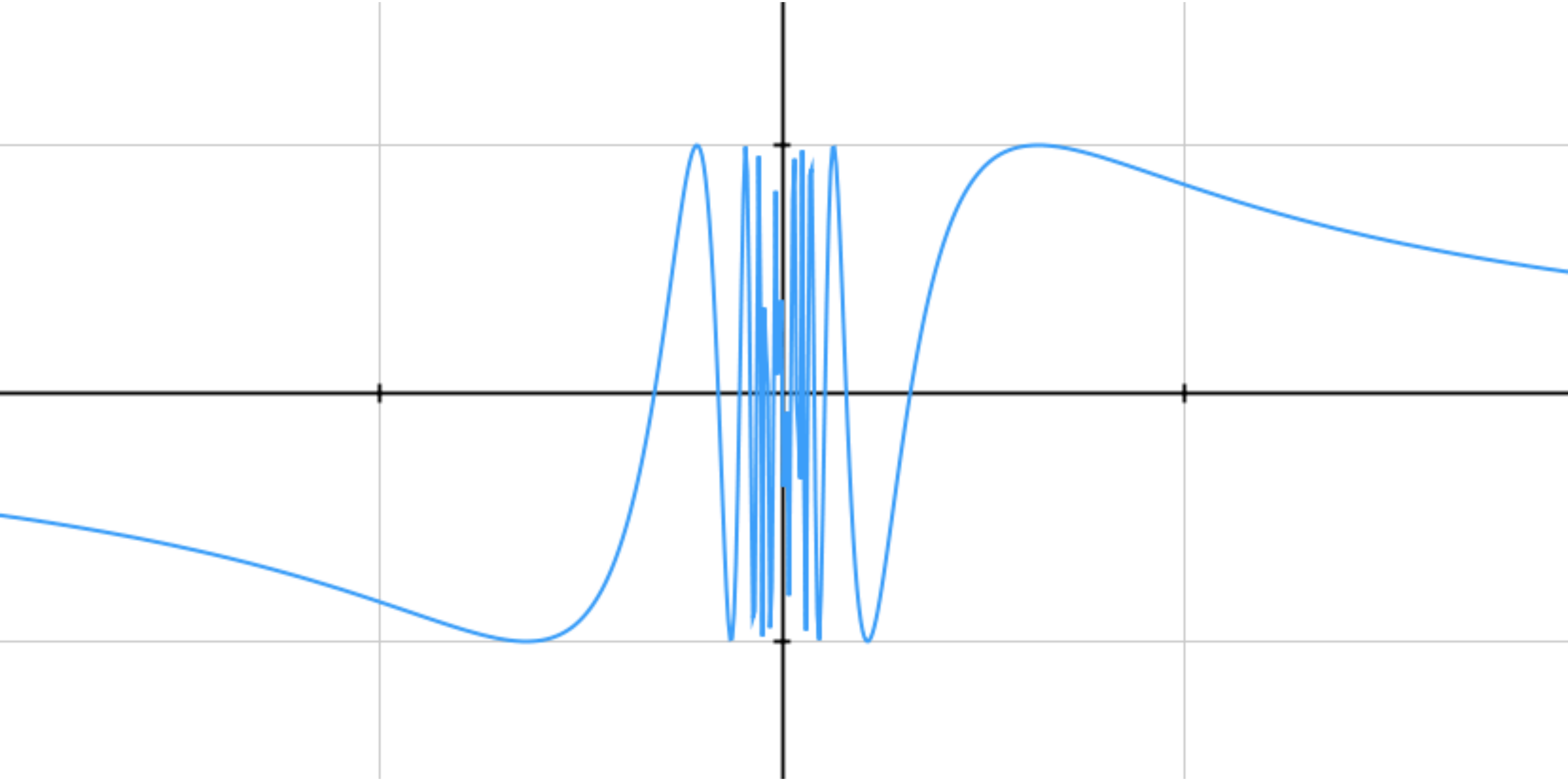
Grothendieck (1984) : for studying the topological properties of natural geometric forms, general topology should be replaced by a tame topology.

You want to discard wild topological phenomena : Cantor sets, Peano curves, but also much more basic phenomena.

Example :

$$\Gamma = \text{graph of } \left(\begin{array}{ccc} \mathbb{R}_{>0} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sin \frac{1}{x} \end{array} \right)$$

Γ is not tame !



Γ is not tame :

- Γ connected but not arc connected
- $\dim \partial\Gamma = \dim \Gamma$
- $\Gamma \cap \mathbb{R}_{-}$ is an infinite collection of points.

Grothendieck' idea of tame topology has been developed axiomatically by model theorists :

o-minimal structures

<u>Definition</u> : a structure expanding $(\mathbb{R}, +, \cdot, <)$ is a collection $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$, called <u>$\mathcal{S}$-definable sets</u>, such that :

1/ algebraic sets of $\mathbb{R}^n$ are in S_n.

2/ S_n is a boolean subalgebra of $\mathcal{P}(\mathbb{R}^n)$

3/ $A \in S_p$, $B \in S_q \Longrightarrow A \times B \in S_{p+q}$

4/ If $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ linear projection and $A \in S_{n+1}$, then $p(A) \in S_n$

A function $f : A \longrightarrow B$ is $\mathcal{S}$-definable if $A, B, \Gamma(f)$ are $\mathcal{S}$-definable.

Prototype : $\mathbb{R}_{alg}$, where the definable sets are the semi-algebraic sets.

Rem : $\mathcal{S}_1, \mathcal{S}_2$ two structures $\Longrightarrow$ $\mathcal{S}_1 \cap \mathcal{S}_2$ is a structure

Hence, given $\mathcal{F} = \{ f_i : \mathbb{R}^n \longrightarrow \mathbb{R} ;\ X_j \subset \mathbb{R}^m \}$, one can define

$\mathbb{R}_{\mathcal{F}}$:= the smallest structure making elements of $\mathcal{F}$ definable.

For instance : $\mathbb{R}_{exp}$, $\mathbb{R}_{sin}$, ...

Facts : • A definable $\Longrightarrow$ $\bar{A}$, $\mathring{A}$, ∂A definable.

• $f : A \longrightarrow B$ definable $\Longrightarrow$ $f(A)$, $f^{-1}(B)$ definable

• $\left.\begin{array}{l} f : A \longrightarrow B \\ g : B \longrightarrow C \end{array}\right\}$ definable $\Longrightarrow g \circ f$ definable

Def : a structure $\mathcal{S}$ is <u>o-minimal</u> if $\underset{\text{definable}}{A \subset \mathbb{R}}$ $\Longrightarrow$ A is a finite union of points and intervals.

(Pillay, Steinhorn, Van den Dries)

Ex: $\mathbb{R}_{alg}$, $\mathbb{R}_{an} = \mathbb{R}\langle f : [-1,1]^n \longrightarrow \mathbb{R}$ real analytic$\rangle$, $\mathbb{R}_{exp}$ $\left(\begin{array}{l} x \longmapsto x^\alpha \quad \alpha \text{ irrational} \\ x \longmapsto e^{-1/x} \end{array} \right)$

Gabrielov

Wilkie (Khovanski)

$\mathbb{R}_{an,exp}$ Miller – Van den Dries

Properties:

Decomposition theorem: Given $A_1, \ldots, A_m \subset \mathbb{R}^n$ definable (in some o-minimal structure)

there exists a finite definable cellular decomposition of $\mathbb{R}^n$ / each A_i is a union of cells.

Cor: A definable $\Rightarrow$

- $|\pi_0(A)| < +\infty$; any connected component of A is definable;
- $\dim \partial A < \dim A$; nice stratification theory;
- any definable $f : X \longrightarrow Y$ is trivializable on $Y = \coprod_{1 \le i \le n} Y_i$

Globalisation: a tame manifold is a topological space M endowed with a finite atlas of charts $\varphi_i : M_i \xrightarrow{\sim} U_i \subset \mathbb{R}^n$ / $U_{ij} := \varphi_j(M_i \cap M_j)$ are definable (in some o-minimal structure) with change of coordinates $\varphi_i \circ \varphi_j^{-1} : U_{ij} \xrightarrow{\sim} U_{ji}$ definable.

Motto: the pathologies of complex analysis are not compatible with tame geometry.

Lemma: Let $f : \Delta^\times \longrightarrow \mathbb{C}$ be holomorphic and definable (in some o-minimal structure). Then 0 is not an essential singularity of f.

Proof: otherwise $\overline{\Gamma(f)} \setminus \Gamma(f) \supset \{0\} \times \mathbb{C}$ (great Picard theorem)

$\Longrightarrow \dim_{\mathbb{R}} \overline{\Gamma(f)} \setminus \Gamma(f) = 2 = \dim \Gamma(f)$: contradiction $\square$

This implies strong algebraization results:

Theor (Peterzil-Starchenko, "o-minimal Chow"): Let $X \underset{\substack{\mathbb{C}\text{-analytic} \\ \text{def}}}{\subset} S$ quasi projective (e.g $S = \mathbb{C}^n$)

Then X is algebraic.

5/ Results

Theorem (Bakker – K. – Tsimerman)

Any period map $\Phi : S^{an} \longrightarrow \Gamma\backslash D$ associated with a (polarizable) $\mathbb{Z}$VHS on S is naturally definable in $\mathbb{R}_{an,exp}$.

Cor (Cattani – Deligne – Kaplan)

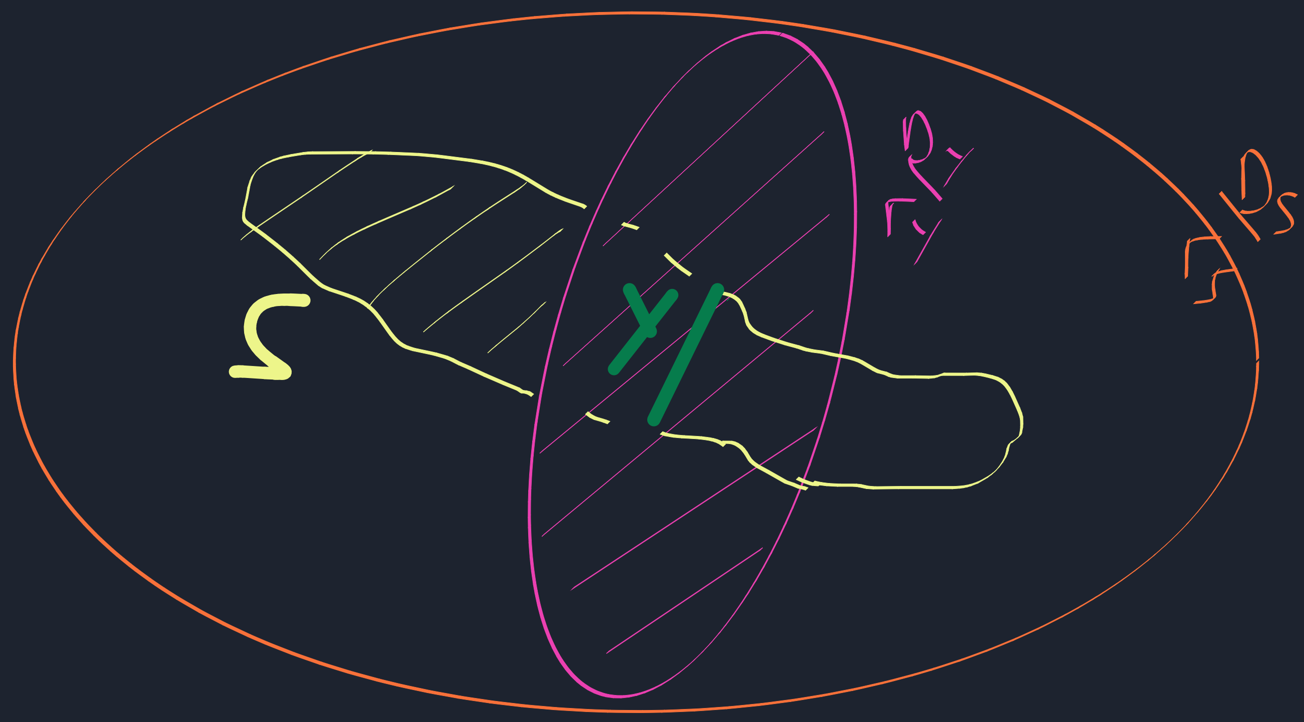
Let $\mathbb{V}$ be a $\mathbb{Z}$VHS on S.

The Hodge locus $HL(S, \mathbb{V}^{\otimes}) = \{ s \in S^{an},\ \mathbb{V}_s \text{ has "exceptional" Hodge tensors} \}$ is a countable union of algebraic subvarieties of S.

Pf:

$$\begin{tikzcd} S^{an} \arrow[r, "\Phi"] & \Gamma\backslash D \\ HL(S,\mathbb{V}^{\otimes}) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \coprod_{\text{countable}} \Gamma'\backslash D' \arrow[u, phantom, "\cup \ \begin{smallmatrix}\mathbb{C}\text{-analytic}\\ \text{definable}\end{smallmatrix}"] \end{tikzcd}$$

$Y \subset S$ is <u>special</u> for $\mathbb{V}$ $\iff$ $Y = \Phi_S^{-1}(\Gamma_Y \backslash D_Y)^\circ$



Y is algebraic by Thm 1 + o-minimal Chow.

$$HL(S, \mathbb{V}) = \bigcup_{\text{countable}} (\text{special subvarieties})$$

$\square$

Thm 1 (K; Otwinowska) Let $\mathbb{V} \longrightarrow S$ ZVHS. Suppose G_S^{ad} is simple.

Let $HL(S, \mathbb{V}^{\otimes})_{pos}$:= union of (strict) special subvarieties whose image under the period map is positive dimensional.

Then $HL(S, \mathbb{V}^{\otimes})_{pos}$ is a finite union of strict special subvarieties of S for $\mathbb{V}$, or it is Zariski-dense in S.

Cor: $S \hookrightarrow A_g$ Hodge generic.

Then either $(S \cap HL(A_g))_{pos}$ is a finite union of special subvarieties of S, or it is Zariski-dense in S.

(Colombo-Pirola; Izadi; Chai): if $\operatorname{codim}_{A_g} S \leq g$ then $HL(S, \mathbb{V}^{\otimes})_{pos}$ is dense in S for the usual topology).